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One-dimensional nondiffracting pulses

Alessandro Ciattoni*
Istituto Nazionale per la Fisica della Materia, UdR, Universitá dell'Aquila, 67010 l'Aquila, Italy and Dipartimento di Fisica, Universitá dell'Aquila, 67010 l'Aquila, Italy

Paolo Di Porto
Dipartimento di Fisica, Universitá dell'Aquila, 67010 l'Aquila, Italy and Istituto Nazionale per la Fisica della Materia, UdR "La Sapienza," 00185 Roma, Italy

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A general expression describing nondiffracting pulses whose transverse profile is a one-dimensional image is presented. The pulse turns out to be expressed as a superposition of two fields, possessing a purely translational dynamics, whose profiles are related to the field distribution on the the waist plane through an Hilbert transformation. The space-time structure of the generally X-shaped pulse is investigated and a simple relation connecting its transverse and the longitudinal widths is established. Specific analytical examples are considered and, in particular, the fundamental one-dimensional X waves are deduced and compared to their two-dimensional counterparts.

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I. INTRODUCTION

The investigation of nondiffracting and, more generally, shape-invariant pulses has attracted a good deal of attention in the last decade. An experimental observation of optical monochromatic diffraction-free beams dates back to Durnin *et al.* [1] and has considerably increased the interest of researchers in the subject of diffraction-free linear propagation. In fact, the possibility of producing monochromatic nondiffracting beam and limited diffraction pulses may find many applications in optical communications [2] and in all those field (e.g., near-field optical microscopy [3–5]) where diffraction hampers the fully exploitation of the directional propagational character of the optical field.

Among the non monochromatic pulses, the most striking and important ones are the limited diffraction pulses or X waves, originally introduced by Lu and Greenleaf [6,7]. These rather exotic fields turn out to be exact solutions of the wave equation and they have the remarkable property of rigidly propagating in vacuum without any distortion [8–12]. Experimentally, X waves have been observed both in acoustics [7] and in optics [13–15]. On the other hand, optical X waves exhibit superluminal features and possess an infinite amount of energy, the first one not being a serious shortcoming since it can be proved that special relativity is not violated [16], while the second is a consequence of the model schematization [17].

A nondiffracting pulse propagating along a given direction is a $(2+1)$-dimensional $[(2+1)\text{D}]$ object since the two transverse coordinates are independent while the longitudinal one and the time appear combined, so that only this combination effectively plays the role of a coordinate. Even if the interest in nondiffracting pulses arises from the fact that they are three-dimensional objects not undergoing the diffractive spreading, there are also situations of practical interest where it is important to consider $(1+1)\text{D}$ nondiffracting pulses. For example, if optical propagation in a slab waveguide is concerned, it is evident that the modal structure of the electromagnetic field affects only the transverse direction parallel to the thickness of the slab, leaving the coordinate along the orthogonal direction free. The part of the electromagnetic field depending on the free coordinate can thus be required to be a one-dimensional nondiffracting pulse.

In the present paper, we investigate $(1+1)\text{D}$ nondiffracting pulses propagating in vacuum. As expected, the reduced dimensionality allows us to fully understand their dynamics. We start from a general representations of $(2+1)\text{D}$ nondiffracting pulses and obtain an expression valid for $(1+1)\text{D}$ pulses, simply imposing that the field distribution on the waist plane effectively depends on a single transverse coordinate. This has the immediate consequence of reducing the dimensionality of the Fourier spectrum of the pulse in the sense that the necessary plane waves belong to two straight lines in the $\mathbf{k}$ space rather than to a cone. As a consequence, we are able to express the pulse as a superposition of two fields whose space-time dependencies are $x-\eta(z-Vt)$ and $x+\eta(z-Vt)$, respectively, x being the only transverse coordinate and η a parameter related to the velocity of the pulse. The relevance of the description stems from the fact that these two fields are simply related to the waist field-distribution through an Hilbert transformation. Exploiting our expression describing one-dimensional nondiffracting pulses, we are able to derive some general properties characterizing their dynamics. First, we prove that the field profile is generally X shaped in the case of bell-shaped distributions of waist field and relate the semiaperture angle to the velocity of the pulse. Furthermore, we show that the transverse and longitudinal widths of the pulse are generally connected by a simple and intuitive relation, as expected because of the strong space-time coupling governing nondiffracting pulses. In order to test these properties, some

*Electronic address: alessandro.ciattoni@aquila.infn.it

analytically workable examples, among which we consider the family of pulses representing the one-dimensional counterpart of the (2+1)D fundamental X waves. Apart from some formal differences, our one-dimensional X waves exhibit by and large the same features of the X waves of Lu and Greenleaf.

II. GENERAL FORMALISM

Let us consider an arbitrary nondiffracting optical pulse, rigidly traveling along the z-axis in vacuum, with speed V. The complex analytic signal $\hat{f}$ of any cartesian component $f = \mathrm{Re}[\hat{f}]$ of the electromagnetic field can be expressed as [17,18]

$$\hat{f}(\mathbf{r}_\perp, Z) = \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{i\eta|\mathbf{k}_\perp|Z} \tilde{f}(\mathbf{k}_\perp), \tag{1}$$

where $\mathbf{r}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$, $\mathbf{k}_\perp = k_x\hat{\mathbf{e}}_x + y_y\hat{\mathbf{e}}_y$, $Z = z - Vt$, $\eta = (V^2/c^2 - 1)^{-1/2}$, c is the speed of light in vacuum and $\tilde{f}(\mathbf{k}_\perp)$ is an arbitrary function. It is easy to check that the pulse in Eq. (1) satisfy the three-dimensional wave equation

$$\nabla^2 \hat{f} = \frac{1}{c^2}\frac{\partial^2 \hat{f}}{\partial t^2}, \tag{2}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. The rigid motion of the pulse formally stems from its dependence on z and t only through Z which is the longitudinal coordinate in a reference frame where the pulse is at rest. The parameter η depends on the speed of the pulse only and it possesses a useful geometrical interpretation. More precisely, from Eq. (1) it is evident that the nondiffracting pulse emerges as a superposition of all the plane waves whose wave vectors $\mathbf{k} = \mathbf{k}_\perp + k_z\hat{\mathbf{e}}_z$ belong to the cone $k_z = \eta|\mathbf{k}_\perp|$ (obviously in the $\mathbf{k}$ space) whose semiaperture angle ψ (called in literature Axicon angle) is given by the relation $\tan\psi = \eta^{-1}$ [19]. It is also worth noting that the pulse of Eq. (1) also satisfies the two-dimensional wave equation

$$\nabla_\perp^2 \hat{f} = \frac{1}{\eta^2}\frac{\partial^2 \hat{f}}{\partial Z^2}, \tag{3}$$

where $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$, which shows how η plays the role of an adimensional speed in the space $(\mathbf{r}_\perp, Z)$. Equation (3) agrees with the general observation of Ref. [8] according to which a N-dimensional nondiffracting pulse satisfies also a (N-1)-dimensional wave equation. In order for $\hat{f}$ to represent a genuine traveling pulse, it is necessary that η is real, which implies $V > c$, thus recovering the well-known superluminality of nondiffracting pulses. Evaluating $\hat{f}$ at $Z=0$ and inverting the obtained Fourier integral we readily get

$$\tilde{f}(\mathbf{k}_\perp) = \frac{1}{(2\pi)^2}\int d^2\mathbf{k}_\perp e^{-i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \hat{f}(\mathbf{r}_\perp, 0), \tag{4}$$

so that $\tilde{f}$ is the two-dimensional Fourier transform of the boundary field $\hat{f}(\mathbf{r}_\perp, 0)$. Equations (1) and (4) allow us to set up a procedure for describing any nondiffracting pulse: In the rest reference, once the field distribution is known at $Z=0$, Eq. (4) gives $\tilde{f}$ and, subsequently, Eq. (1) gives the pulse for every Z. It is worth noting that here there is no constrain on the boundary field distribution $\hat{f}(\mathbf{r}_\perp, 0)$, except the obvious requirement that both the integrals in Eqs. (1) and (4) converge. In this sense, any image at $Z=0$ can be stored in the waist of the pulse thus allowing a sort of diffraction-free transmission of arbitrary two-dimensional images. Since any image stored in $\hat{f}(\mathbf{r}_\perp, 0)$ travels without suffering diffractive spreading, the transmission is characterized by an unlimited resolution. From a physical point of view, the arbitrariness of $\hat{f}(\mathbf{r}_\perp, 0)$ is a consequence of the mentioned conical spectral structure of nondiffracting pulses. More specifically, each Fourier component of $\hat{f}(\mathbf{r}_\perp, 0)$ (labeled by $\mathbf{k}_\perp$), excites only the plane wave whose frequency allows its wave vector to belong to the cone, preventing, for example, the appearance of evanescent waves.

Exploiting the arbitrariness of the boundary distribution $\hat{f}(\mathbf{r}_\perp, 0)$, we want now to investigate the class of nondiffracting pulses exhibiting translation invariance along a given transverse direction (one-dimensional nondiffracting pulses, ONP), that is pulses whose transverse profile is, at any plane $Z = \bar{Z}$, a one-dimensional image. In particular, $\hat{f}(\mathbf{r}_\perp, 0)$ must admit a translational invariance direction and, since a fully rotational invariance around the longitudinal z- direction is present, we can only require $\hat{f}(\mathbf{r}_\perp, 0)$ not to depend on y, or

$$\hat{f}(\mathbf{r}_\perp, 0) = \hat{f}(x, 0). \tag{5}$$

Following the above propagation scheme, we substitute Eq. (5) into Eq. (4), thus obtaining

$$\tilde{f}(\mathbf{k}_\perp) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx e^{-ik_x x}\hat{f}(x,0)\,\delta(k_y) \equiv \tilde{f}(k_x)\,\delta(k_y), \tag{6}$$

where $\delta(k_y)$ is the Dirac delta-function and $\tilde{f}(k_x)$ is the one-dimensional Fourier transform of the function $\hat{f}(x,0)$. The nondiffracting pulse corresponding to the boundary distribution of Eq. (5) is obtained after substituting Eq. (6) into Eq. (1), which yields

$$\hat{f}(x, Z) = \int_{-\infty}^{+\infty} dk_x e^{ik_x x} e^{i\eta|k_x|Z} \tilde{f}(k_x). \tag{7}$$

This is the most general expression describing an ONP or a pulse which is at the same time nondiffracting and one dimensional. Note that $\hat{f}(x, Z)$ depends on x and Z in a quite similar way through two exponentials, and that both factors are affected by the function $\tilde{f}(k_x)$ at the same time. Since $Z = z - Vt$, the space and temporal behaviors of the pulse are closely related, resulting in a strong space-time coupling which is typical of nondiffacting pulses [18].

The pulse in Eq. (7) contains only the plane waves [each one being eigen-solutions of Eq. (3) after $\nabla_\perp^2$ is replaced by ∂_x^2], whose wave vectors belong to the two straight lines $k_z = \eta|k_x|$ which are the intersection between the cone $k_z = \eta|\mathbf{k}_\perp|$ and the plane $k_y = 0$. The main features characterizing the pulse can be pointed out by representing it as a superposition of two fields, each containing the plane waves belonging to one of the two lines $k_z=\eta k_x$ and $k_z=-\eta k_x$ (with $k_z>0$). After some manipulation, Eq. (7) can be rewritten as

$$\hat{f}(x,Z) = F^{(+)}(x-\eta Z) + F^{(-)}(x+\eta Z), \tag{8}$$

where we introduced the functions

$$F^{(\pm)}(\xi) = \int_{-\infty}^{+\infty} dk_x e^{ik_x\xi}\theta(\mp k_x)\tilde{f}(k_x), \tag{9}$$

where $\theta(x)$ is the usual step-function. Note that, as a consequence of Eq. (3), the pulse in Eq. (7) must satisfy the one-dimensional wave equation

$$\frac{\partial^2 \hat{f}}{\partial x^2} = \frac{1}{\eta^2}\frac{\partial^2 \hat{f}}{\partial Z^2} \tag{10}$$

and that, consequently, Eq. (8) coincides with the D'Alembertian solution of this equation. Equation (8) contains a basic decomposition since it recasts the problem of evaluating an ONP into that of evaluating the pair of functions defined in Eq. (9). From this equation it is evident that $F^{(+)}(\xi)$ and $F^{(-)}(\xi)$ are obtained from the function $\hat{f}(x,0)$ [whose Fourier transform is $\tilde{f}(k_x)$] by suppressing the positive $k_x>0$ and negative $k_x<0$ part of the spectrum, respectively. Introducing the definition of $\tilde{f}(k_x)$ from Eq. (6) into Eq. (9) and performing the integral over k_x we obtain, after introducing suitable terms $\exp(\pm\epsilon k_x)$ for ensuring convergence

$$F^{(\pm)}(\xi) = \pm\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx \frac{\hat{f}(x,0)}{(\xi - x)\mp i\epsilon}, \tag{11}$$

where the limit $\epsilon \to 0^+$ is understood to be taken after the evaluation of the integral. Exploiting the well-known relation $(x\pm i\epsilon)^{-1} = \mathrm{P}(1/x)\mp i\pi\delta(x)$ where P indicates the Cauchy principal value, Eq. (11) finally yields

$$F^{(\pm)}(\xi) = \frac{1}{2}[\hat{f}(\xi,0)\mp i\mathcal{H}\hat{f}(\xi)], \tag{12}$$

where

$$\mathcal{H}\hat{f}(\xi) = \frac{1}{\pi}P\int_{-\infty}^{+\infty} dx \frac{\hat{f}(\xi,0)}{\xi - x} \tag{13}$$

is the Hilbert transform of $\hat{f}(\xi,0)$. Equation (12) relates the functions $F^{(\pm)}$ to the boundary distribution of the pulse and to its Hilbert transform. The propagation scheme for ONP is thus as follows: From the knowledge of boundary distribution $\hat{f}(x,0)$ of the pulse, it is sufficient to evaluate its Hilbert transform so that Eq. (8), with the aid of Eq. (12), gives the pulse for every Z.

III. SPATIAL CHARACTERIZATION OF ONP

Equation (8) states that the ONP is the superposition of two fields whose profiles, at any transverse plane $Z=\overline{Z}$, coincide with their own distributions at $Z=0$ translated, along the x-axis, from $x=0$ to $x=\eta\overline{Z}$ and $x=-\eta\overline{Z}$, respectively. Suppose that the boundary distribution $\hat{f}(x,0)$ is localized around the origin $x=0$. It is easy to show that the Hilbert transform $\mathcal{H}\hat{f}(\xi)$ is also localized around the origin so that the functions $F^{(\pm)}(\xi)$ are not negligible only in a given neighbor of $\xi=0$. Therefore, in the plane xZ, the fields $F^{(+)}(x-\eta Z)$ and $F^{(-)}(x+\eta Z)$ globally resides around the straight lines $x=\pm\eta Z$ (at least asymptotically for Z large enough), respectively, so that (see Fig. 1) the overall shape of the pulse is that of a letter X whose semiaperture angle ϕ is given by $\tan\phi=\eta$ (one-dimensional X wave). Since η is directly related to V, it is evident that the speed of the pulse can be experimentally measured simply measuring ϕ. Taking into account the relation connecting the Axicon angle ψ to η we deduce that $\phi=\pi/2-\psi$. Around the origin of the plane xZ, the pulse generally exhibits a peak since the overlap region between $F^{(+)}(x-\eta Z)$ and $F^{(-)}(x+\eta Z)$ is there not negligible.

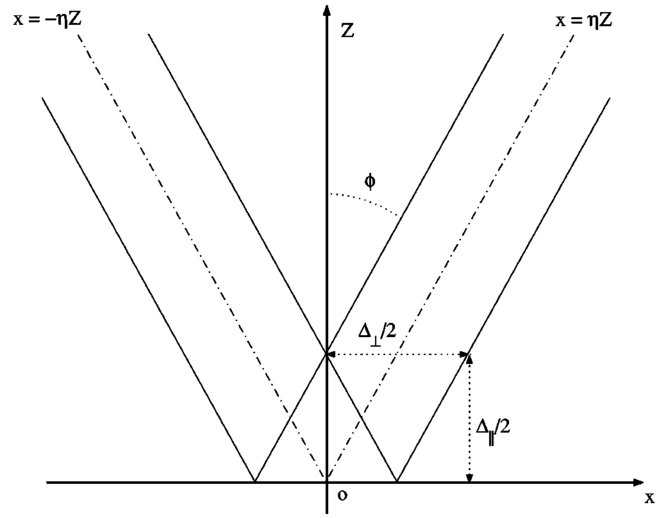

FIG. 1. Typical geometry of a ONP.

In the case of even boundary distribution, $\hat{f}(x,0)=\hat{f}(-x,0)$ the spatial properties of the central peak can be simply described. Let us consider the transverse and longitudinal autocorrelation widths [20] given by

$$\Delta_\perp = \frac{\left|\int_{-\infty}^{+\infty} dx\hat{f}(x,0)\right|^2}{\int_{-\infty}^{+\infty} dx|\hat{f}(x,0)|^2}, \quad \Delta_\parallel = \frac{\left|\int_{-\infty}^{+\infty} dZ\hat{f}(0,Z)\right|^2}{\int_{-\infty}^{+\infty} dZ|\hat{f}(0,Z)|^2}, \tag{14}$$

respectively. The former is an effective width of the boundary distribution $\hat{f}(x,0)$ whereas the latter is a measure of the longitudinal width of the on-axis $(x=0)$ pulse $\hat{f}(0,Z)$. We have chosen these kinds of widths since it can be shown (see the Appendix) that, for our nondiffracting pulses, the relation

$$\Delta_{\parallel} = \frac{1}{2\eta}\Delta_{\perp} \tag{15}$$

is satisfied. This is a remarkable relation joining the longitudinal length of the central peak to its lateral width and it is an evident manifestation of the strong space-time coupling characterizing nondiffracting pulse. In fact, the quantity $\Delta\tau = \Delta_{\parallel}/V$ may be regarded as the time duration of the central peak and Eq. (15) states that it is uniquely fixed by the transverse lateral extension of $\hat{f}(x,0)$ and by the velocity V of the pulse. Equation (15) has a simple and intuitive graphical interpretation. To this end, first note that it is easy to prove that, introducing the widths of the functions $F^{(\pm)}(\xi)$, one has $\Delta^{(\pm)} = \Delta_{\perp}/2$ (see the Appendix). Referring to Fig. 1, it is evident that the central peak of the pulse, along the z-axis, practically fall off when the overlap region between the two arms vanishes, i.e., when $\Delta_{\perp}/4 = \eta\Delta_{\parallel}/2$, which coincide with Eq. (15).

IV. ANALYTICAL EXAMPLES

In order to specialize the above general discussion, we now consider some examples admitting analytical solutions. Consider the ONP whose boundary distribution is

$$\hat{f}(x,0) = f_0 \frac{\sin(Kx)}{Kx}, \tag{16}$$

where f_0 and K are constants, that is the well-known sinc distribution. The function $\hat{f}(x,0)$ is peaked at the origin $x=0$ and its width, according to Eqs. (14), is $\Delta_{\perp} = \pi/K$. It is straightforward to prove that [20] $\mathcal{H}\hat{f}(\xi) = f_0[1 - \cos(K\xi)]/(K\xi)$, so that the functions defined in Eq. (12) can be expressed as

$$F^{(\pm)}(\xi) = f_0 e^{\mp iK\xi/2} \frac{\sin\left(\frac{K\xi}{2}\right)}{K\xi} \tag{17}$$

(note that their moduli are peaked at $\xi=0$). Inserting Eq. (17) into Eq. (8) we readily get

$$\hat{f}(x,Z) = f_0 \frac{\eta KZ + e^{i\eta KZ}[iKx\sin(Kx) - \eta KZ\cos(Kx)]}{iK^2(x^2 - \eta^2 Z^2)}. \tag{18}$$

In Fig. 2, we plot, for $\eta=2.18$ (corresponding to $V=1.1c$), the normalized field $\mathrm{Re}[\hat{f}/f_0]$ from Eq. (18). Note that, as expected, the field corresponding to the boundary sinc distribution (at $Z=0$) breaks, for increasing Z, into two arms lying on the two lines $x=\eta Z$ and $x=-\eta Z$, respectively. From the second of Eqs. (14) and (18) we obtain a longitudinal width $\Delta_{\parallel} = \pi/(2\eta K)$, which, compared with $\Delta_{\perp} = \pi/K$, confirms Eq. (15) in this particular case.

As a further analytical example, let us consider the family of ONP whose spectrum is

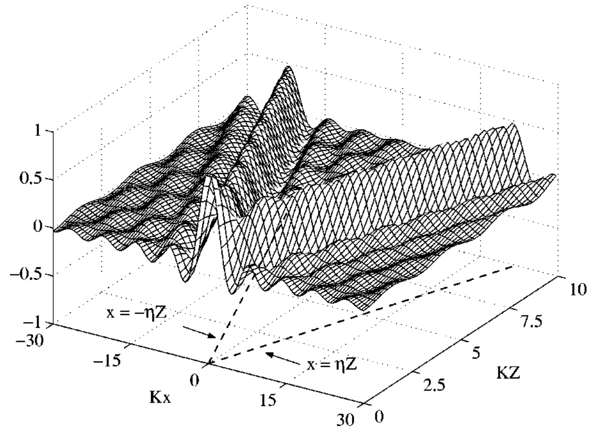

FIG. 2. Plot of the normalized field $\mathrm{Re}[\hat{f}/f_0]$ of a ONP pulse whose boundary distribution is the sin **c** function of Eq. (16), as a function of Kx and KZ. The speed of the pulse chosen for this plot is $V=1.1c$, corresponding to $\eta=2.18$.

$$\tilde{f}_n(k_x) = f_0 s^{n+1}|k_x|^n e^{-s|k_x|}, \tag{19}$$

where $n=0,1,\ldots$, and f_0 and s are constants. The interest in this example emerges from the fact that these ONPs are the one-dimensional counterpart of the well-known two-dimensional X waves of Lu and Greenleaf [7], so that they may be termed one-dimensional X waves. Inserting Eq. (19) into Eqs. (9) we obtain

$$F_n^{(\pm)}(\xi) = f_0 \frac{n!}{\left(1 \pm i\frac{\xi}{s}\right)^{n+1}}. \tag{20}$$

Note that both $F_n^{(\pm)}$ are peaked at the origin, s being a parameter setting their widths. Even if any one-dimensional X wave $\hat{f}_n(x,Z)$ of this family can be directly obtained from Eq. (8) with the aid of Eq. (20), it is convenient here to use the relation

$$\hat{f}_n(x,Z) = \left(\frac{s}{i\eta}\frac{\partial}{\partial Z}\right)^n \hat{f}_0(x,Z), \tag{21}$$

where

$$\hat{f}_0(x,Z) = f_0 \frac{2\left(1 - \frac{i\eta Z}{s}\right)}{\left(1 - \frac{i\eta Z}{s}\right)^2 + \left(\frac{x}{s}\right)^2}, \tag{22}$$

as it can be shown by substituting Eq. (19) into Eq. (7). In analogy with the two-dimensional case [11,19], Eq. (21) shows that the whole family of one-dimensional X waves is obtained by repeatedly differentiating the first pulse $\hat{f}_0(x,Z)$. In Fig. 3 we plot the normalized field $\mathrm{Re}[\hat{f}_n/f_0]$ of the $n=0$ and $n=1$ one-dimensional X waves, whose speed is $V=1.1c$, corresponding to $\eta=2.18$. The essential features of these fields are quite similar to those of the sinc X wave depicted in Fig. 1. Comparing the one-dimensional X waves of Eq. (21) with the two-dimensional ones [11,19], we note a

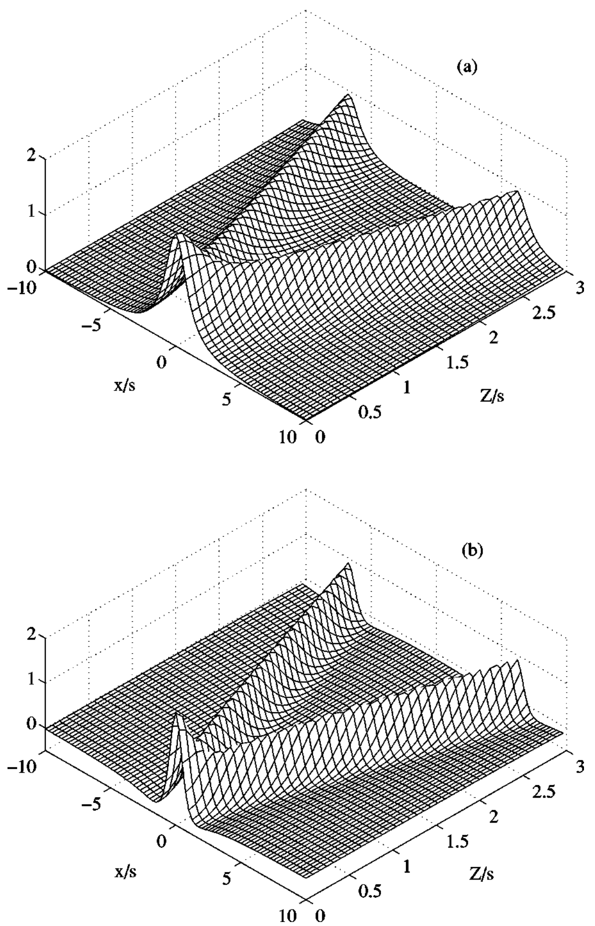

FIG. 3. Plot of the normalized field $\mathrm{Re}[\hat{f}_n/f_0]$ pertaining to the one-dimensional X waves (a) $n=0$ and (b) $n=1$ as a function of x/s and Z/s. The speed of the pulse chosen for this plot is $V=1.1c$, corresponding to $\eta=2.18$.

remarkable resemblance between their structures. Conversely, the most striking difference is the absence of half-integer powers of n in the denominator of the formers. Moreover, $\hat{f}_0 \approx x^{-2}$ for $|x| \to +\infty$ whereas the first two-dimensional X wave shows the asymptotical behavior of $(x^2+y^2)^{-2}$; this implies that a one-dimensional X wave transversally decays faster than the two-dimensional X wave of the same order.

V. CONCLUSIONS

Exploiting a formalism describing an arbitrary nondiffracting pulse propagating in vacuum, we have investigated one-dimensional nondiffracting pulses (ONP), whose transverse profile is a one-dimensional image at any transverse plane. We have deduced a general expression showing that an ONP is the superposition of two fields $F^{(\pm)}$, depending on space and time only through the combinations $x \mp \eta(z-Vt)$, respectively, and simply related to the boundary field distribution $\hat{f}(x,0)$ by means of an Hilbert transform. The obtained decomposition has allowed us to predict the major ONP features, such as the characteristic X shape of the field profile and the relation between the transverse and longitudinal widths of the pulse (a relation having its origin in the strong space-time coupling proper of nondiffracting pulses). In addition, we have considered two analytical examples where the above-mentioned properties can be checked. The second of this example is of some interest in the frame of shape-invariant beams, since it represents the one-dimensional counterpart of the well-known family of fundamental X waves.

APPENDIX: TRANSVERSE AND LONGITUDINAL WIDTHS

The widths defined in Eq. (14) can be manipulated by exploiting Eq. (7). Substituting the former in the expression for $\Delta_\perp$ of Eq. (14), it is straightforward to prove that

$$\Delta_\perp = 2\pi \frac{|\tilde{f}(0)|^2}{\int_{-\infty}^{+\infty} dk_x |\tilde{f}(k_x)|^2}, \tag{A1}$$

where use has been made of the Parseval theorem for changing the denominator. Less straightforward is the manipulation of $\Delta_\parallel$; after substituting Eq. (7) into its expression of Eq. (14), we obtain

$$\Delta_\parallel = \frac{\left| \frac{2}{\eta} \int_{-\infty}^{+\infty} dk_x \tilde{f}(k_x) \int_0^{+\infty} dZ e^{ik_x Z} \right|^2}{\frac{2\pi}{\eta} \int_{-\infty}^{+\infty} dk_x \tilde{f}(k_x) \int_{-\infty}^{+\infty} dk_x' \tilde{f}^*(k_x') \delta(|k_x| - |k_x'|)}, \tag{A2}$$

where the integral representation of the Dirac delta function has been exploited in the denominator. The integral over Z in the numerator can be handled by introducing a convergence factor $\exp(-\epsilon Z)$ (the limit $\epsilon \to 0^+$ being understood to be taken at the end of calculations), so that

$$\int_0^{+\infty} dZ e^{-(\epsilon - ik_x)Z} = \frac{i}{k + i\epsilon} = \pi\delta(k) + i\mathrm{P}\frac{1}{k}, \tag{A3}$$

where P indicates the Cauchy principal value and we have exploited the well-known relation $(x \pm i\epsilon)^{-1} = \mathrm{P}(1/x) \mp i\pi\delta(x)$. Taking into account the identity

$$\delta(|k_x| - |k_x'|) = [\theta(k_x)\theta(k_x') + \theta(-k_x)\theta(-k_x')]\delta(k_x - k_x') + [\theta(-k_x)\theta(k_x') + \theta(k_x)\theta(-k_x')]\delta(k_x + k_x') \tag{A4}$$

and Eq. (A3), Eq.(A2) becomes

$$\Delta_\parallel = \frac{2\pi}{\eta} \frac{\left| \tilde{f}(0) + \frac{i}{\pi} P \int_{-\infty}^{+\infty} dk_x \frac{1}{k_x} \tilde{f}(k_x) \right|^2}{\int_{-\infty}^{+\infty} dk_x |\tilde{f}(k_x)|^2 + \int_{-\infty}^{+\infty} dk_x \tilde{f}(k_x) \tilde{f}^*(-k_x)}. \tag{A5}$$

In the case of even boundary condition, $\hat{f}(x,0) = \hat{f}(-x,0)$, we have $\tilde{f}(k_x) = \tilde{f}(-k_x)$ so that Eq. (A5) yields

$$\Delta_{\|} = \frac{\pi}{\eta} \frac{|\tilde{f}(0)|^2}{\int_{-\infty}^{+\infty} dk_x |\tilde{f}(k_x)|^2}. \qquad (A6)$$

Comparing Eq. (A6) and Eq. (A1), we obtain Eq. (15). Let us now evaluate the widths of the functions $F^{\pm}(\xi)$, defined as

$$\Delta^{(\pm)} = \frac{\left| \int_{-\infty}^{+\infty} d\xi F^{(\pm)}(\xi) \right|^2}{\int_{-\infty}^{+\infty} d\xi |F^{(\pm)}(\xi)|^2}. \qquad (A7)$$

Inserting Eq. (9) into Eq. (A7) we obtain

$$\Delta^{(\pm)} = \pi \frac{|\tilde{f}(0)|^2}{2\int_{-\infty}^{+\infty} \theta(\mp k_x) |\tilde{f}(k_x)|^2}, \qquad (A8)$$

where we have exploited the relation $\theta(0)=1/2$. In the case of even boundary condition [for which $\tilde{f}(k_x)=\tilde{f}(-k_x)$] we obtain from Eqs. (A8) and (A1) the relation

$$\Delta^{(\pm)} = \frac{1}{2}\Delta_{\perp}. \qquad (A9)$$

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